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UNIPOTENT CHARACTERS OF THE EVEN ORTHOGONAL GROUPS OVER A FINITE FIELD

BY

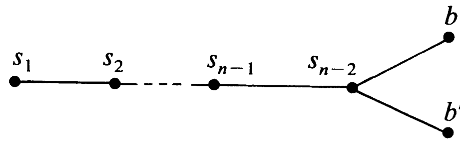
GEORGE LUSZTIG[1]

ABSTRACT. The characters of unipotent representations of a simple algebraic group over F_q of type $\neq D_n$ on any regular semisimple element are explicitly known for large q. This paper deals with the remaining case: type D_n.

The purpose of this paper is to give explicit formulas for the character of the unipotent representations of the special orthogonal group of a quadratic form of even dimension over a finite field F_q on any regular semisimple element, provided that q is sufficiently large. The methods used in this paper are those of **[4]**, where the case of symplectic and odd orthogonal groups was considered. To avoid repetitions, I have only given proofs for the results which differ essentially from those in **[4]**.

1. Characters of Weyl groups and Hecke algebras of type D_n.

1.1. Let D_n $(n \geqslant 2)$ be the Coxeter group with diagram



(Here $s_1, s_2, \ldots, s_{n-2}, b, b'$ are the simple reflections.) For $n = 2$, this is the group generated by the commuting involutions b, b'. We make the convention that D_1 is the group with 1 element. We may regard D_n $(n \geqslant 1)$ as a subgroup of index 2 of the group $\hat{D}_n$ generated by D_n and an element φ of order 2 which commutes with all s_i $(1 \leqslant i \leqslant n-2)$ and satisfies $\varphi b \varphi = b'$ (if $n \geqslant 2$).

1.2. The group $\hat{D}_n$ is isomorphic in two ways with the Coxeter group of type B_n (denoted W_n in **[4]**): One choice of simple reflections is $\{s_1, s_2, \ldots, s_{n-2}, b, \varphi\}$; the other is $\{s_1, s_2, \ldots, s_{n-2}, b', \varphi\}$. Each of these two choices gives rise to an isomorphism $\hat{D}_n \approx W_n$ of Coxeter systems. We may thus parametrize the irreducible $\mathbf{Q}[\hat{D}_n]$-modules by ordered pairs σ_1, σ_2 of irreducible representations of the symmetric groups $\mathfrak{S}_k$, $\mathfrak{S}_l$ $(k + l = n)$, as in **[4, 2.1]** and this parametrization will be independent of the choice of simple reflections for $\hat{D}_n$ (the two choices are conjugate by φ). It will be convenient to parametrize the irreducible $\mathbf{Q}[\hat{D}_n]$-modules by ordered pairs $\binom{S}{T}$ where S, T are finite subsets of $\{0, 1, 2, \ldots\}$ defined as follows: if σ_1, σ_2

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correspond to the partitions $0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_m$, $0 \leqslant \beta_1 \leqslant \beta_2 \leqslant \cdots \leqslant \beta_m$ of k, l respectively, we set $S = \{\alpha_1, \alpha_2 + 1, \ldots, \alpha_m + m - 1\}$, $T = \{\beta_1, \beta_2 + 1, \ldots, \beta_m + m - 1\}$. Here, m can be increased at our will, so that $\binom{S}{T}$ should be regarded as being equivalent to

$$\begin{pmatrix} 0 \cup (S+1) \\ 0 \cup (T+1) \end{pmatrix}$$

and also to pairs obtained from $\binom{S}{T}$ by iterating this "shift" operation. Thus, we have a 1-1 correspondence

$$\begin{pmatrix} S \\ T \end{pmatrix} \leftrightarrow \begin{bmatrix} S \\ T \end{bmatrix}$$

between the set

$$F_n = \left\{ \begin{pmatrix} S \\ T \end{pmatrix} \,\middle|\, \text{ordered pair with } S, T \subset \{0,1,2,\ldots\}, |S| = |T| = m, \right.$$
$$\left. \sum_{\lambda_i \in S} \lambda_i + \sum_{\mu_i \in T} \mu_i = n + m^2 - m, \text{up to shift} \right\}$$

and the set of irreducible $\mathbf{Q}[\hat{D}_n]$-modules (up to isomorphism). Here $\left[\begin{smallmatrix} S \\ T \end{smallmatrix}\right]$ is the $\mathbf{Q}[\hat{D}_n]$-module corresponding to $\binom{S}{T}$. This parametrization is more appropriate for the study of groups of type D_n than that in [**4**, 2.1]. It follows from the definition that

$$\begin{bmatrix} S \\ T \end{bmatrix} \otimes \chi = \begin{bmatrix} T \\ S \end{bmatrix}, \tag{1.2.1}$$

where $\chi\colon \hat{D}_n \to \{\pm 1\}$ is the homomorphism whose kernel is D_n.

1.3. Let w_0 be the longest element of the Coxeter group D_n. Then $\varphi^n w_0$ is the unique nontrivial central element of $\hat{D}_n$. It acts on the irreducible $\mathbf{Q}[\hat{D}_n]$-module $\left[\begin{smallmatrix} S \\ T \end{smallmatrix}\right]$ as multiplication by the scalar

$$(-1)^{\sum_{i=1}^{m}(\mu_i - i + 1)}$$

where $T = \{\mu_1, \mu_2, \ldots, \mu_m\}$. This follows immediately from [**4**, 2.2].

1.4. Let sign′ be the unique homomorphism $\hat{D}_n \to \{\pm 1\}$ such that $\text{sign}'(\varphi) = 1$ and sign′ restricted to D_n is the sign character of the Coxeter group D_n. Let $\binom{S}{T} \in F_n$; assume that it is represented by sets S, T with $|S| = |T| = m$. Choose an integer t which is greater than or equal to any entry of S and T. Let $\bar{S} = \{t - i \mid 0 \leqslant i \leqslant t, i \notin S\}$, $\bar{T} = \{t - i \mid 0 \leqslant i \leqslant t, i \notin T\}$. Then $\binom{\bar{S}}{\bar{T}}$ is independent of the choice of t (up to shift); it defines an element of F_n, and we have

$$\begin{bmatrix} S \\ T \end{bmatrix} \otimes \text{sign}' = \begin{bmatrix} \bar{S} \\ \bar{T} \end{bmatrix} \tag{1.4.1}$$

as representations of $\hat{D}_n$. This follows from [**4**, 2.4] together with (1.2.1).

1.5. Let H be the Hecke algebra (with coefficients in $\mathbf{Q}[u^{1/2}, u^{-1/2}]$) corresponding to the Coxeter group D_n; here $u^{1/2}$ is an indeterminate. Let $\hat{H}$ be the algebra over $\mathbf{Q}[u^{1/2}, u^{-1/2}]$ generated by H and by an element T_φ with the relations $T_\varphi^2 = 1$, $T_\varphi T_w T_\varphi = T_{\varphi w \varphi}$ $(w \in D_n)$, where T_w $(w \in D_n)$ are the canonical basis elements of H.

Thus, H is a subalgebra of $\hat{H}$ which, under the specialization $u^{1/2} \to 1$, becomes the subalgebra $\mathbf{Q}[D_n]$ of $\mathbf{Q}[\hat{D}_n]$.

Let E be an irreducible $\mathbf{Q}[\hat{D}_n]$-module whose restriction $E \mid D_n$ to D_n is irreducible. The construction in [**4**, 1.1] with $W = D_n$ associates to $E \mid D_n$ an irreducible H-module $\tilde{E} = (M_C \otimes_{\mathbf{Q}} E)^{D_n}$ (where $C \subset D_n$ is the two-sided cell of D_n corresponding to $E \mid D_n$). Note that $\tilde{E}$ is free as a $\mathbf{Q}[u^{1/2}, u^{-1/2}]$-module. It has a natural involution which takes an element $\Sigma_{z \in C} e_z \otimes v_z$ to $\Sigma_{z \in C} e_{\varphi z \varphi} \otimes \varphi(v_z)$. (Here e_z, $z \in C$, are the basis elements of M_C, see [**4**, 1.1].) This involution of $\tilde{E}$ is denoted T_φ. Together with the H-module structure on $\tilde{E}$, it defines an $\hat{H}$-module structure on $\tilde{E}$.

We have, as in [**4**, 1.2],

$$\text{(1.5.1)} \qquad \operatorname{Tr}\left(T_x T_\varphi^i, \tilde{E}\right) \in \mathbf{Z}[u^{1/2}] \qquad (i = 0, 1; x \in D_n).$$

Moreover, with the notations in [**4**, 1.2] we have

(1.5.2)

$$\operatorname{Tr}\left(T_x T_\varphi^i, \tilde{E}\right) = |D_n|^{-1} \sum_{w \in D_n} \sum_{\substack{z, z' \in C \\ \varphi^i z \varphi^i \underset{L}{\sim} z' \underset{R}{\sim} z}} \tau\left(T_x C_{\varphi^i z \varphi^i} D_{z'^{-1}}\right) \tau\left(D_{z^{-1}} C_{z'} T_w\right)\big|_{u=1} \operatorname{Tr}\left(w\varphi^i, E\right)$$

$$(i = 0, 1; x \in D_n).$$

The proof is the same as that of [**4**, 1.3]. This implies, as in [**4**, 1.6], that

$$\text{(1.5.3)} \qquad \operatorname{Tr}\left(T_{x^{-1}}^{-1} T_\varphi^i, \tilde{E}\right) = \overline{\operatorname{Tr}\left(T_x T_\varphi^i, \tilde{E}\right)} \qquad (i = 0, 1; x \in D_n),$$

where $a \to \bar{a}$ is the involution of the ring $\mathbf{Q}[u^{1/2}, u^{-1/2}]$ such that $\overline{u^{j/2}} = u^{-j/2}$. The following result is proved in the same way as [**4**, 1.7]:

$$\text{(1.5.4)} \qquad \operatorname{Tr}\left(T_x T_\varphi^i, \tilde{E}\right) = \operatorname{Tr}\left(T_{x^{-1}} T_\varphi^i, \tilde{E}\right) \qquad (i = 0, 1; x \in D_n).$$

Let $\operatorname{Dim}(\tilde{E})$ be the "formal dimension" of the H-module $\tilde{E}$ (an element of $\mathbf{Q}[u]$). We have the identity ($i = 0$ or 1):

$$\text{(1.5.5)} \qquad \sum_{x \in D_n} u^{-l(x)} \operatorname{Tr}\left(T_x T_\varphi^i, \tilde{E}\right) \operatorname{Tr}\left(T_{x^{-1}} T_\varphi^i, \tilde{E}\right) = \frac{\Sigma_{w \in D_n} u^{l(w)}}{\operatorname{Dim}(\tilde{E})} \cdot \dim(E)$$

(where l is the length function on D_n).

For $i = 0$, this is a special case of [**4**, (1.8.1)]. We now prove it for $i = 1$. The left-hand side of (1.5.5) is the trace of the $\mathbf{Q}(u^{1/2})$-linear map

$$h \to \sum_{x \in D_n} u^{-l(x)} T_x T_\varphi^i h T_\varphi^i T_{x^{-1}} : J \to J$$

where J is the minimal 2-sided ideal of $H \otimes \mathbf{Q}(u^{1/2})$ corresponding to the irreducible representation $\tilde{E} \otimes \mathbf{Q}(u^{1/2})$. But one checks easily that $\Sigma_{x \in D_n} u^{-l(x)} T_x h' T_{x^{-1}}$ is in the centre of $H \otimes \mathbf{Q}(u^{1/2})$ for all $h' \in H \otimes \mathbf{Q}(u^{1/2})$. Hence our linear map has image consisting of scalar multiples of a single vector v, which spans the intersection of J with the centre. Hence its trace is equal to λ where $\lambda v = \Sigma_{x \in D_n} u^{-l(x)} T_x T_\varphi^i v T_\varphi^i T_{x^{-1}}$. But v must be also in the centre of $\hat{H}$ (since $\tilde{E}$ is an $\hat{H}$-module); thus $T_\varphi^i v T_\varphi^i = v$ and so λ is independent of i, as required.

Let us define integers $A(E)$, $a(E)$ (cf. [4, 1.8]) as follows: $A(E)$ is the degree of the polynomial (in u) $\mathrm{Dim}(\tilde{E})$ and $u^{a(E)}$ is the largest power of u dividing this polynomial. (These should not be confused with the integers $A(E)$, $a(E)$ defined using a Coxeter group structure of type B_n on $\hat{D}_n$, which are different.) Using (1.5.4), (1.5.5), as in [4, 1.9], we deduce that for $i = 0$ and 1, and $x \in D_n$, we have

$$(1.5.6)\quad \mathrm{Tr}\left(T_x T_\varphi^i, \tilde{E}\right) = \begin{cases} \text{constant} \cdot u^{(l(x)-a(E))/2} + \text{higher powers of } u^{1/2}, \\ \text{constant} \cdot u^{(l(x)-A(E)+\nu)/2} + \text{lower powers of } u^{1/2}. \end{cases}$$

(Here $\nu = n^2 - n$ is the length of $w_0 \in D_n$.) As in [4, 1.11] we deduce that

$$(1.5.7)\quad \begin{gathered} T_{w_0} T_\varphi^n \text{ acts on } \tilde{E} \text{ as scalar multiplication by } \pm u^{\nu-(a(E)+A(E))/2} \text{ and} \\ \mathrm{Tr}\left(T_{xw_0} T_\varphi^i, \tilde{E}\right) = \pm \overline{\mathrm{Tr}\left(T_x T_\varphi^{i+n}, \tilde{E}\right)} \end{gathered}$$

where the sign ± 1 is the scalar by which $w_0\varphi^n$ acts on E ($i = 0, 1$; $x \in D_n$). We have

$$(1.5.8)\quad \mathrm{Tr}\left(T_x T_\varphi^i, (E \otimes \mathrm{sign}')^{\tilde{}}\right) = (-u)^{l(x)} \mathrm{Tr}\left(T_x T_\varphi^i, \tilde{E}\right) \qquad (i = 0, 1;\ x \in D_n)$$

where sign′ is defined as in 1.4. (Compare [4, 1.13].) We now define

$$(1.5.9)\qquad \mathrm{Tr}\left(T_x T_\varphi^i, \tilde{E}; h/2\right) \in \mathbf{Z} \qquad (i = 0, 1;\ x \in D_n)$$

to be the coefficient of $u^{h/2}$ in $\mathrm{Tr}(T_x T_\varphi^i, \tilde{E})$, see (1.5.1). For each $x \in D_n$, we define $\alpha_x, \mathcal{A}_x \in \mathfrak{R}(D_n)$ (= group of virtual representations of D_n) by the formulas [4, (1.15.1), (1.15.2)], with $W = D_n$. We also define two virtual $\hat{D}_n$-representations $\alpha_{x\varphi}$, $\mathcal{A}_{x\varphi}$ by the formulas:

$$(1.5.10)\qquad \alpha_{x\varphi} = (-1)^{l(x)} \sum_E \mathrm{Tr}\left(T_x T_\varphi, \tilde{E};\ \frac{l(x) - a(E)}{2}\right) E;$$

$$(1.5.11)\qquad \mathcal{A}_{x\varphi} = \sum_E \mathrm{Tr}\left(T_x T_\varphi \tilde{E};\ \frac{l(x) - A(E) + \nu}{2}\right) E;$$

both sums are over all irreducible $\hat{D}_n$-modules E which remain irreducible on restriction to D_n, $l(x)$ is the length in the Coxeter group D_n and $\nu = n^2 - n$. Thus, $\alpha_{x\varphi}$ and $A_{x\varphi}$ belong to $\mathfrak{R}(\hat{D}_n)^-$, the group of virtual representations of $\hat{D}_n$ which are anti-invariant under the Z-linear involution ι which interchanges any two distinct irreducible representations of $\hat{D}_n$ which are the same as D_n-modules, and keeps fixed any irreducible representation of $\hat{D}_n$ which is reducible as a D_n-module. We have (cf. (1.5.8), [4, 1.14, 1.16])

$$(1.5.12)\qquad \mathcal{A}_x = \alpha_x \otimes (\mathrm{sign}' \,|\, D_n), \qquad \mathcal{A}_{x\varphi} = \alpha_{x\varphi} \otimes \mathrm{sign}'.$$

In the case where n is even, so that w_0 is in the centre of D_n of $\hat{D}_n$, we have defined in [4, 1.17] a $\mathbf{Z}$-linear involution ζ of the group $\mathfrak{R}(D_n)$: it sends an irreducible D_n-module E to $\pm E$ where ± 1 is the sign by which w_0 acts on E. We can also define a $\mathbf{Z}$-linear involution of the group of virtual representations of $\hat{D}_n$ which sends an irreducible $\hat{D}_n$-module E to $\pm E$ where ± 1 is the sign by which w_0 acts on E; we denote again by ζ, the restriction of this involution to the group $\mathfrak{R}(\hat{D}_n)^-$. (It is mapped into itself by ζ.)

In the case where n is odd, so that $w_0\varphi$ is in the centre of $\hat{D}_n$ we define two **Z**-linear bijections (inverse to each other)

$$\mathcal{R}(D_n) \underset{\zeta}{\overset{\zeta}{\rightleftarrows}} \mathcal{R}(\hat{D}_n)^-$$

as follows. If E is an irreducible D_n-module, let E' (resp. E'') be the unique $\hat{D}_n$-module which becomes E on restriction to D_n and is such that $w_0\varphi$ acts on it as $+1$ (resp. as -1). We set $\zeta(E) = E' - E''$. This gives one of the bijections ζ. The other one is its inverse.

We have

$$(1.5.13) \qquad \mathcal{C}_{x\varphi^i w_0} = (-1)^{l(x)}\zeta(\alpha_{x\varphi^{i+n}}) \qquad (i = 0, 1;\ x \in D_n)$$

where ζ has four possible meanings (explained above) according to whether i and n are even or odd.

1.6. The irreducible $\mathbf{Q}[D_n]$-modules can be parametrized as follows. If E is such a module and if E can be extended to a $\hat{D}_n$-module $[{S \atop T}]$, then we associate to E the *unordered* pair $({S \atop T})$, and we still denote $E = [{S \atop T}]$; note that the other $\hat{D}_n$-module extending E is $[{T \atop S}]$ (cf. (1.2.1)), so the unordered pair S, T is well-defined (up to shift); it is a symbol of rank n and defect 0 (up to shift) (see [**2**, §3]). If E cannot be extended to a $\hat{D}_n$-module, then the representation induced by E to $\hat{D}_n$ is irreducible, of form $[{S \atop S}]$. We associate to E the symbol $({S \atop S})$, up to shift; in this case, the symbol does not determine the isomorphism class of E uniquely: there are two irreducible representations of D_n corresponding to it. Thus we have a correspondence

$$\begin{bmatrix} S \\ T \end{bmatrix} \to \begin{pmatrix} S \\ T \end{pmatrix}$$

between irreducible representations of D_n and symbols of rank n and defect 0 (up to shift) which is 1-1 except that to symbols of form $({S \atop S})$ (said to be degenerate) correspond two representations of D_n (also said to be degenerate).

According to [**4**, 1.8], there are well-defined invariants $a(E)$, $A(E)$, γ_E of an irreducible D_n-module E. (When E extends to an irreducible representation E' of $\hat{D}_n$, then $a(E)$, $A(E)$ are the same as $a(E')$, $A(E')$, as defined in 1.5.) If E corresponds to

$$\begin{pmatrix} S \\ T \end{pmatrix} = \begin{pmatrix} \lambda_1, \ldots, \lambda_m \\ \mu_1, \ldots, \mu_m \end{pmatrix},$$

we have

$$(1.6.1) \qquad \begin{aligned} a(E) = \sum_{i<j} \inf(\lambda_i, \lambda_j) + \sum_{i<j} \inf(\mu_i, \mu_j) \\ + \sum_{i,j} \inf(\lambda_i, \mu_j) - \frac{m}{6}(m-1)(4m-5), \end{aligned}$$

$$(1.6.2) \qquad \gamma_E = \begin{cases} 2^{-(d-1)}, & \text{if } d \geq 1, \\ 1, & \text{if } d = 0, \end{cases}$$

where $2d = (S \cup T) - (S \cap T)$ is the number of "singles" in $({S \atop T})$.

1.7. The proofs of the statements in this section are similar to those of [**4**, 2.7, 2.8]. Assume that $n \geqslant 2$ and that $r + s = n$ is a partition of n such that $n \geqslant 1$, $s \geqslant 1$. We identify $\mathfrak{S}_r \times \hat{D}_s$ with the subgroup of $\hat{D}_n$ generated by $s_1, \ldots, s_{r-1}, s_{r+1}, \ldots, s_{n-2}, b, b', \varphi$ and $\mathfrak{S}_r \times D_s$ with the corresponding subgroup of D_n. (Thus $\mathfrak{S}_r \times D_s$ is the intersection of $\mathfrak{S}_r \times \hat{D}_s$ with D_n.)

Let $\varepsilon(r)$ be the sign representation of $\mathfrak{S}_r$. We wish to describe the D_n-module $J_{\mathfrak{S}_r \times D_s}^{D_n}(\varepsilon(r) \otimes [{S \atop T}])$ (see [**4**, 1.19]) where $[{S \atop T}]$ is an irreducible representation of D_s with $S \neq T$. We may assume that $2m \geqslant r$. This D_n-module is $[{S' \atop T'}]$ or $[{S' \atop T'}] + [{S'' \atop T''}]$, where $({S' \atop T'})$, $({S'' \atop T''})$ are obtained from $({S \atop T})$ by increasing each of the n largest entries in S, T by 1 and leaving the others unchanged. This process can be done either in a unique way or in two distinct ways (see [**4**, 2.6]) and it leads either to a unique symbol $({S' \atop T'})$ with $S' \neq T'$ or to two symbols $({S' \atop T'})$, $({S'' \atop T''})$ with $S' \neq T'$, $S'' \neq T''$. In the case where $S = T$, the same process leads again either to a unique symbol $({S' \atop T'})$ with $S' = T'$ (if r is even) or to two symbols $({S' \atop T'})$, $({S'' \atop T''})$ with $S' \neq T'$, $S'' \neq T''$, $S' = T''$, $S'' = T'$ (if r is odd); we have

$$J_{\mathfrak{S}_r \times D_s}^{D_n}\left(\varepsilon(r) \otimes \begin{bmatrix} S \\ T \end{bmatrix}\right) = \begin{bmatrix} S' \\ T' \end{bmatrix}. \tag{1.7.1}$$

Here $[{S \atop T}]$ can have 2 possible meanings (since $S = T$), but if r is odd, the formula holds with both these meanings: $[{S' \atop T'}]$ is well defined. If r is even, $[{S \atop T}]$ and $[{S' \atop T'}]$ both have 2 possible meanings, and our formula is ambiguous. We will make it unambiguous as follows. One can label the two irreducible D_n-modules corresponding to a symbol $({S' \atop T'})$ with $S' = T'$ as $[{S' \atop T'}]_b$, $[{S' \atop T'}]_{b'}$ in such a way that (1.7.1) with r, s, n even can be written in the unambiguous form:

$$\begin{cases} J_{\mathfrak{S}_r \times D_s}^{D_n}\left(\varepsilon(r) \otimes \begin{bmatrix} S \\ T \end{bmatrix}_b\right) = \begin{bmatrix} S' \\ T' \end{bmatrix}_b, \\ J_{\mathfrak{S}_r \times D_s}^{D_n}\left(\varepsilon(r) \otimes \begin{bmatrix} S \\ T \end{bmatrix}_{b'}\right) = \begin{bmatrix} S' \\ T' \end{bmatrix}_{b'}. \end{cases} \tag{1.7.2}$$

(Note that b, b' in D_n and b, b' in D_s are in natural 1-1 correspondence, $n \geqslant s \geqslant 2$.)

This labelling is unique if we require that it also has the following property. Let $\mathfrak{S}_n^{(b)}$ be the subgroup of D_n ($n \geqslant 2$) generated by all simple reflections $\neq b'$. (It is a symmetric group.) Similarly, we define $\mathfrak{S}_n^{(b')} = \varphi \mathfrak{S}_n^{(b)} \varphi$. Then

$$J_{\mathfrak{S}_n^{(b)}}^{D_n}(\varepsilon(n)) = \begin{cases} \begin{bmatrix} 1 & 2 & \cdots & \frac{n}{2} \\ 1 & 2 & \cdots & \frac{n}{2} \end{bmatrix}_b & \text{if } n \text{ is even}, \\ \begin{bmatrix} 1 & 2 & 3 & \cdots & \frac{n+1}{2} \\ 0 & 2 & 3 & \cdots & \frac{n+1}{2} \end{bmatrix} & \text{if } n \text{ is odd}, \end{cases} \tag{1.7.3}$$

and the same holds with b replaced by b'. Note also that

$$J_{\mathfrak{S}_r \times D_s}^{D_n}\left(\varepsilon(r) \otimes \alpha_z^{(D_s)}\right) = \alpha_{w_0^{(r)} z}^{(D_n)} \qquad (n = r + s, r \geqslant 1, s \geqslant 1) \tag{1.7.4}$$

where $w_0^{(r)}$ is the longest element in $\mathfrak{S}_r$; $z \in D_s$, $\alpha_z^{(D_s)}$ means α_z with respect to D_s; and $\alpha_{w_0^{(r)} z}^{(D_n)}$ means $\alpha_{w_0^{(r)} z}$ with respect to D_n; see [**4**, 1.20, 1.21, 1.22]. Moreover,

$$J_{\mathfrak{S}_n^{(b)}}^{D_n}(\varepsilon(n)) = \alpha_{w_{0,b}^{(n)}}^{(D_n)}, \tag{1.7.5}$$

where $w_{0,b}^{(n)}$ is the longest element in $\mathfrak{S}_n^{(b)}$; the same holds with b replaced by b'.

1.8. Assume that $n \geqslant 2$ and that $r + s = n$ is a partition of n such that $r \geqslant 1$, $s \geqslant 1$. Consider an irreducible representation $[\frac{S}{T}]$ of $\hat{D}_s$ with $S \neq T$ (S, T is an ordered pair). We consider the $\hat{D}_n$-module $J_{\mathfrak{S}_r \times D_s}^{D_n}(\varepsilon(r) \otimes [\frac{S}{T}])$, defined in the same way as in [4, 1.19] except that the invariant $a(E)$ is defined as in 1.5 (using the Coxeter group structure of D_s, D_n). This $\hat{D}_n$-module is again $[\frac{S'}{T'}]$ or $[\frac{S'}{T'}] + [\frac{S''}{T''}]$, where these are obtained from S, T by increasing each of the r largest entries in S, T by 1 and leaving the others unchanged; we assume here, as we may, that $|S| = |T| \geqslant r$. (Note that S', T' and S'', T'' are ordered pairs.) The map $J_{\mathfrak{S}_r \times D_s}^{D_n}$ extends by **Z**-linearity to a map

$$\mathcal{R}(\hat{D}_s)^- \to \mathcal{R}(\hat{D}_n)^- .$$

We have

$$J_{\mathfrak{S}_r \times \hat{D}_s}^{\hat{D}_n}\left(\alpha_{z\varphi}^{(\hat{D}_s)}\right) = \alpha_{w_0^{(r)} z\varphi}^{(\hat{D}_n)} \qquad (z \in D_s),$$

where the elements φ of $\hat{D}_s$, $\hat{D}_n$ are identified via the natural inclusion $\hat{D}_s \subset \hat{D}_n$, and the upper-scripts $(\hat{D}_s)$, $(\hat{D}_n)$ have a meaning similar to that in 1.7.

1.9. Let

$$Z = \begin{pmatrix} z_2, z_4, \ldots, z_{2m} \\ z_1, z_3, \ldots, z_{2m-1} \end{pmatrix}$$

be a tableau consisting of integers z_i satisfying $z_1 < z_3 < \cdots < z_{2m-1}$, $z_2 < z_4 < \cdots < z_{2m}$, $0 \leqslant z_1 \leqslant z_2 \leqslant z_3 \leqslant \cdots \leqslant z_{2m-1} \leqslant z_{2m}$, $\sum_{i=1}^{2m} z_i = n + m^2 - m$ ($n \geqslant 1$). We say that Z is degenerate if $z_1 = z_2$, $z_3 = z_4, \ldots, z_{2m-1} = z_{2m}$; otherwise, it is nondegenerate. We shall regard Z as being equivalent to its shifts (see §1.2). We say that Z is a special symbol of rank n and defect 0. If Z is nondegenerate, we denote by $[Z]$ the corresponding irreducible representation $\hat{D}_n$ (see §1.2). From (1.6.1) we see that $a[Z] \equiv \sum_{i=1}^{m}(z_{2i-1} - i + 1) \pmod 2$. Using 1.3, it follows that $w_0\varphi^n$ acts on $[Z]$ as multiplication by $(-1)^{a[Z]}$.

1.10. Assume that Z is nondegenerate. Then it has $2d > 0$ "singles" (= entries appearing in exactly one row of Z). Let Φ be an arrangement of the $2d$ "singles" in Z into d disjoint pairs such that each pair in Φ contains one single in the first row of Z and one in the second row of Z. We define what it means for Φ to be an *admissible arrangement* for Z in the same way as in [4, 2.11], by induction on $d \geqslant 1$. (Compare [5].) Here we must start the induction with the case $d = 1$ in which case the unique arrangement is, by definition, admissible. For example, $Z = \binom{1\ 3}{0\ 2}$ has two admissible arrangements: one of them consists of the pairs $(0, 1)$ and $(2, 3)$; the other one consists of the pairs $(1, 2)$, $(0, 3)$. As another example,

$$Z = \begin{pmatrix} 1 & 3 & 5 \\ 0 & 2 & 1 \end{pmatrix}$$

has five admissible arrangements:

the first one consists of $(0, 1)$, $(2, 3)$, $(4, 5)$,

the second one consists of $(1, 2)$, $(3, 4)$, $(0, 5)$,

the third one consists of $(1, 2)$, $(4, 5)$, $(0, 3)$,

the fourth one consists of $(0,1)$, $(3,4)$, $(2,5)$,
the fifth one consists of $(2,3)$, $(1,4)$, $(0,5)$.

Now let Φ be an admissible arrangement for Z. If Ψ is any subset of Φ, we denote by Ψ^* (resp. Ψ_*) the set of singles in the first row (resp. second row) of Z which appear in a pair of Ψ. Let Z_2 be the set of elements which appear in both rows of Z.

Let $\hat{\Phi}$ be a subset of Φ. We define

$$\underline{c}(Z,\Phi,\hat{\Phi}) = \sum_{\Psi\subset\Phi} (-1)^{e(\Psi)} \begin{bmatrix} Z_2 \amalg \Psi_* \amalg (\Phi-\Psi)^* \\ Z_2 \amalg \Psi^* \amalg (\Phi-\Psi)_* \end{bmatrix}$$

where $e(\Psi) = |\hat{\Phi}^* \cap \Psi^*|$. This sum is interpreted as follows. If $|\hat{\Phi}|$ is even, then $(-1)^{e(\Psi)} = (-1)^{e(\Phi-\Psi)}$ and we take the sum over equivalence classes of subsets $\Psi \subset \Phi$ (Ψ is equivalent to $\Phi - \Psi$); the expression in square brackets is interpreted as an irreducible representation of D_n (it is the same for Ψ and for $\Phi - \Psi$). Thus the sum has 2^{d-1} terms; it is a virtual representation of D_n. If $|\hat{\Phi}|$ is odd, then $(-1)^{e(\Psi)} = -(-1)^{e(\Phi-\Psi)}$ and our sum with 2^d terms is an ι-anti-invariant virtual representation of $\hat{D}_n$.

In any case $\underline{c}(Z,\Phi,\hat{\Phi})$ is said to be a (nondegenerate) *virtual cell*.

1.11. If we define $\bar{Z}$ (with Z nondegenerate) as in 1.4, with respect to an integer $t \geq z_{2m}$, we get again a nondegenerate special symbol of rank n and defect 0. We have a 1-1 correspondence $z \leftrightarrow t - z$ between the singles in Z and the singles in $\bar{Z}$. Given $\hat{\Phi} \subset \Phi$ for Z as in 1.10 (Φ admissible), we can thus transport them to $\bar{Z}$ and we get an admissible arrangement $\bar{\Phi}$ for $\bar{Z}$ and a subset $\bar{\hat{\Phi}}$ of $\bar{\Phi}$. We have

$$\underline{c}(Z,\Phi,\hat{\Phi}) \otimes \text{sign}' = \underline{c}(\bar{Z},\bar{\Phi},\bar{\hat{\Phi}}) \tag{1.11.1}$$

(cf. (1.4.1)).

1.12. We now consider the subgroup $\mathfrak{S}_r \times \hat{D}_s$ of $\hat{D}_n$ and the corresponding subgroup $\mathfrak{S}_r \times D_s$ of D_n ($r+s=n$, $r \geq 1$, $s \geq 1$). Let

$$Z' = \begin{pmatrix} z'_2, z'_4, \ldots, z'_{2m} \\ z'_1, z'_3, \ldots, z'_{2m-1} \end{pmatrix}$$

be a nondegenerate special symbol of rank s and defect 0; we shall assume, as we may, that $2m \geq r$. We associate to Z' the nondegenerate special symbol of rank n and defect 0

$$Z = \begin{pmatrix} z_2, z_4, \ldots, z_{2m} \\ z_1, z_3, \ldots, z_{2m-1} \end{pmatrix}$$

defined by $z_i = z'_i$ ($1 \leq i \leq 2m-r$), $z_i = z'_i + 1$ ($2m+1-r \leq i \leq 2m$). Suppose that we are given an admissible arrangement Φ' for Z' and a subset $\hat{\Phi}'$ of Φ'. We transport these to Z using the natural bijection $z'_i \leftrightarrow z_i$ between Z' and Z. In the case where $r = 2m$ or $r \leq 2m-1$ and $z'_{2m-1-r} < z'_{2m-r}$ (so that Z, Z' have the same number of singles) we thus get an admissible arrangement Φ for Z and a subset $\hat{\Phi}$ of Φ. In the case where $z'_{2m-1-r} = z'_{2m-r}$ (so that Z has two new singles in addition to those coming from Z'), the set of pairs in Z coming from those in Φ' together with the new pair (z_{2m-1-r}, z_{2m-r}) form an admissible arrangement Φ for Z. It has a subset $\hat{\Phi}$ corresponding to the pairs in $\hat{\Phi}'$ (the new pair is not in $\hat{\Phi}$). Using now the results in 1.7 we see that

$$J^{D_n}_{\mathfrak{S}_r \times D_s}(\varepsilon(r) \otimes \underline{c}(Z', \Phi', \hat{\Phi}')) = \underline{c}(Z, \Phi, \hat{\Phi})$$

if $|\hat{\Phi}'|$ is even and that the same formula holds when $|\hat{\Phi}'|$ is odd, provided that we replace D_s, D_n by $\hat{D}_s$, $\hat{D}_n$.

1.13. Let Z, Φ, $\hat{\Phi}$ be as in 1.11. Let Φ_1 be the set of pairs (z_i, z_j) in Φ such that $z_i + z_j$ is odd. Let $\Phi_2 \subset \Phi$ be defined by $\Phi_2 = (\hat{\Phi} \cup \Phi_1) - (\hat{\Phi} \cap \Phi_1)$. We have

$$(-1)^{a[Z]}\zeta(\underline{c}(Z, \Phi, \hat{\Phi})) = \underline{c}(Z, \Phi, \Phi_2)$$

where ζ is defined in 1.5. The proof is the same as that of [**4**, 2.18].

1.14. Now let

$$Z = \begin{pmatrix} z_1, z_3, z_5, \ldots, z_{2m-1} \\ z_1, z_3, z_5, \ldots, z_{2m-1} \end{pmatrix}$$

be a degenerate special symbol of rank n and defect 0 (n is necessarily even). It gives rise to two irreducible representations $[Z]_b$, $[Z]_{b'}$ of D_n (see 1.7). They are said to be degenerate virtual cells. The tensor product of $[Z]_b$ or $[Z]_{b'}$ with sign$'\,|\,D_n$ is $[\bar{Z}]_b$ or $[\bar{Z}]_{b'}$ (not necessarily in this order), where $\bar{Z}$ is defined as in 1.4, with respect to an integer $t \geqslant z_{2m-1}$.

The behaviour of $[Z]_b$ under the operation J is described in 1.7. We have

$$\zeta([Z]_b) = (-1)^i[Z]_b, \qquad \zeta([Z]_{b'}) = (-1)^i[Z_{b'}]$$

where $i = a([Z]_b) = a([Z]_{b'})$.

1.15. We now define, by induction on $n \geqslant 1$, a certain set of involutions $\hat{\Omega}_n \subset \hat{D}_n$. For $n = 1$, we take $\hat{\Omega}_1 = \hat{D}_1$. Assume now that $n \geqslant 2$ and that $\hat{\Omega}_s \subset \hat{D}_s$ is already defined for $1 \leqslant s < n$. We say that $w \in \hat{D}_n$ is in $\hat{\Omega}_n$ if

(a) there exist a partition $n = r + s$ ($s \geqslant 1$, $r \geqslant 1$) and an element $z \in \hat{\Omega}_s \subset \hat{D}_s$ such that $w = w_0^{(r)} \cdot z \in \mathfrak{S}_r \times \hat{D}_s \subset \hat{D}_n$, or $w = (w_0\varphi^n)(w_0^{(r)} \cdot z)$, or

(b) w is the longest element in $\mathfrak{S}_n^{(b)} \subset D_n$ or in $\mathfrak{S}_n^{(b')} \subset D_n$ (see 1.7), or

(c) $(w_0\varphi^n)w$ is the longest element in $\mathfrak{S}_n^{(b)}$ or in $\mathfrak{S}_n^{(b')}$. (Here w_0 is the longest element in D_n and $w_0^{(r)}$ is the longest element in $\mathfrak{S}_r$.)

Using the results in 1.5–1.15 just as in [**4**, 2.20], one proves the following two results. (Although Propositions 1.16, 1.17 are stated separately, their proofs must be carried out simultaneously.)

PROPOSITION 1.16. *Assume that $n \geqslant 1$. The following 3 sets of virtual representations of D_n coincide*:

(a) $\{\alpha_w \mid w \in \hat{\Omega}_n \cap D_n\}$,

(b) $\{\mathcal{C}_w \mid w \in \hat{\Omega}_n \cap D_n\}$,

(c) *the set of virtual cells $\underline{c}(Z, \Phi, \hat{\Phi})$ with Z special nondegenerate of rank n, defect 0, $|\hat{\Phi}|$ even, and the virtual cells $[Z]_b$, $[Z]_{b'}$ with Z special degenerate of rank n, defect 0. (The latter exist only for even n.) Moreover, if $\underline{c}(Z, \Phi, \hat{\Phi})$ or $[Z]_b$ or $[Z]_{b'}$ is equal to $\mathcal{C}_w = \alpha_{w'}$ ($w, w' \in \hat{\Omega}_n \cap D_n$) then*

$$a[Z] \equiv l(w') \pmod 2, \qquad \nu - A[Z] \equiv l(w) \pmod 2.$$

PROPOSITION 1.17. *Assume that* $n \geqslant 1$. *The following* 3 *sets of* ι*-anti-invariant virtual representations of* $\hat{D}_n$ *coincide*:

(a) $\{\alpha_w \mid w \in \hat{\Omega}_n, w \notin D_n\}$,

(b) $\{\mathcal{C}_w \mid w \in \hat{\Omega}_n, w \notin D_n\}$,

(c) *the set of virtual cells* $\underline{c}(Z, \Phi, \hat{\Phi})$ *with* Z *special nondegenerate of rank* n, *defect* 0, *and with* $|\hat{\Phi}|$ *odd. Moreover, if* $\underline{c}(Z, \Phi, \hat{\Phi}) = \mathcal{C}_w = \alpha_{w'}$ $(w, w' \in \hat{\Omega}_n,\ w, w' \notin D_n)$ *then*

$$a[Z] \equiv l(w'\varphi) \pmod 2, \qquad \nu - A[Z] \equiv l(w\varphi) \pmod 2.$$

Just as in [4, 2.21] we obtain from 1.16, 1.17 the following

COROLLARY 1.18. *Assume that* $n \geqslant 1$. *Let* $\underline{c}$ *be a virtual cell as in* 1.16(c) *and let* $w \in D_n$ *be such that* $\underline{c} = \mathcal{C}_w$. *Then*

$$\underline{c} = \sum \operatorname{Tr}\big(T_w, \tilde{E}; (l(w) - A(E) + \nu)/2\big)E$$

(*sum over all irreducible* D_n*-modules* E). *If* $\underline{c}$ *is a virtual cell as in* 1.17(c) *and* $w \in D_n$ *is such that* $\underline{c} = \mathcal{C}_{w\varphi}$, *then*

$$\underline{c} = \sum \operatorname{Tr}\big(T_w T_\varphi, \tilde{E}; (l(w) - A(E) + \nu)/2\big)E$$

(*sum over all irreducible nondegenerate* $\hat{D}_n$*-modules* E). *In both cases, all irreducible components* E *of* $\underline{c}$ *have the same* $a(E)$ *and the same* $A(E)$. *The common value of* $a(E)$ *is called* $a(\underline{c})$ *and the common value of* $A(E)$ *is called* $A(\underline{c})$.

LEMMA 1.19. *Let* E *be an irreducible, nondegenerate representation of* $\hat{D}_n$. *There exist a nondegenerate special symbol* Z *of rank* n *and defect* 0, *an admissible arrangement* Φ *for* Z *and a subset* $\Psi \subset \Phi$ *such that with the notation of* 1.10, *we have*

$$E = \begin{bmatrix} Z_2 \amalg \Psi_* \amalg (\Phi - \Psi)^* \\ Z_2 \amalg \Psi^* \amalg (\Phi - \Psi)_* \end{bmatrix}.$$

We have

$$E \mid D_n = 2^{-(d-1)} \sum_{\substack{\hat{\Phi} \subset \Phi \\ |\hat{\Phi}| = \text{even}}} (-1)^{e'(\hat{\Phi})} \underline{c}(Z, \Phi, \hat{\Phi}),$$

$$E - \iota(E) = 2^{-(d-1)} \sum_{\substack{\hat{\Phi} \subset \Phi \\ |\hat{\Phi}| = \text{odd}}} (-1)^{e'(\hat{\Phi})} \underline{c}(Z, \Phi, \hat{\Phi}),$$

where $2d$ *is the number of singles in* Z *and*

$$e'(\hat{\Phi}) = |\hat{\Phi}^* \cap \Psi^*|.$$

The proof is the same as that of Lemma 2.22 in [4].

2. A combinatorial result.

2.1. Let Y be a vector space of dimension $2d - 1$ $(d \geqslant 1)$ over the field F_2, endowed with a basis $e_1, e_2, \ldots, e_{2d-1}$ and with a symplectic form $(\ ,\ )$: $Y \times Y \to F_2$

such that

$$(e_i, e_j) = \begin{cases} 1 & \text{if } |i-j| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

This form is singular; its radical is spanned by the vector $\varepsilon = e_1 + e_3 + e_5 + \cdots + e_{2d-1}$.

If $d \geqslant 2$, we consider for each i, $1 \leqslant i \leqslant 2d-1$, the vector space

$$Y_i = \{y \in Y \mid (e_i, y) = 0\}/\{0, e_i\}.$$

It inherits from Y a symplectic form and a basis:

$$\begin{array}{ll} e_1, e_2, \ldots, e_{i-2}, e_{i-1} + e_{i+1}, e_{i+2}, \ldots, e_{2d-1} & \text{if } i \neq 1, 2d-1, \\ e_3, e_4, \ldots, e_{2d-1} & \text{if } i = 1, \\ e_1, e_2, \ldots, e_{2d-3} & \text{if } i = 2d-1. \end{array}$$

This basis has the same relation to the symplectic form on Y_i as the basis $e_1, \ldots, e_{2d-1}$ had with (,) on Y.

2.2. We define by induction on d, a family $\mathfrak{F}(Y)$ of maximal isotropic subspaces of V, depending on the given basis (e_i) of Y. If $d = 1$, $\mathfrak{F}(Y)$ consists of Y. Assume now that $d \geqslant 2$ and that $\mathfrak{F}(Y_i)$ has been already defined for $1 \leqslant i \leqslant 2d-1$ (with respect to the basis of Y_i described above). By definition, a d-dimensional subspace C of Y is in $\mathfrak{F}(Y)$ if and only if there exists i, $1 \leqslant i \leqslant 2d-1$, such that $e_i \in C$ and such that the image of C under the natural map $\{y \in Y \mid (e_i, y) = 0\} \to Y_i$ is in $\mathfrak{F}(Y_i)$. For example, if $d = 2$, $\mathfrak{F}(Y)$ consists of two subspaces: the first is spanned by e_1, e_3 and the second is spanned by $e_2, e_1 + e_3$. If $d = 3$, $\mathfrak{F}(Y)$ consists of five subspaces: the first one is spanned by e_1, e_3, e_5, the second one is spanned by $e_1, e_4, e_3 + e_5$, the third is spanned by $e_1 + e_3, e_2, e_5$, the fourth one is spanned by $e_2, e_4, e_1 + e_3 + e_5$, and the fifth is spanned by $e_3, e_2 + e_4, e_1 + e_5$.

The following properties of a subspace $C \in \mathfrak{F}(Y)$ follow easily from the definitions.

(2.2.1) If $e_i \in C$, then the image of C in Y_i is in $\mathfrak{F}(Y_i)$.

(2.2.2) If $e_1 \in C$ and $d \geqslant 2$, then some e_i, $i \geqslant 2$, is in C.

(2.2.3) If $e_{2d-1} \in C$ and $d \geqslant 2$, then some e_i, $i \leqslant 2d-2$, is in C.

Let $\tilde{Y}$ be the set of vectors in Y which lie in some subspace in $\mathfrak{F}(Y_i)$. We have

LEMMA 2.3. *Assume that $d \geqslant 2$. Given i, $1 \leqslant i \leqslant 2d-1$, and two elements y, $y' \in \tilde{Y}$ such that $(y, e_i) = (y', e_i) = 1$, there exist a sequence of elements $y = y_1, y_2, \ldots, y_m = y'$ in $\tilde{Y}$ and a sequence of subspaces $C_1, C_2, \ldots, C_{m-1}$ in $\mathfrak{F}(Y)$ such that y_h, $y_{h+1} \in C_h$ $(1 \leqslant h \leqslant m-1)$ and $(y_h, e_i) = 1$ $(1 \leqslant h \leqslant m)$.*

PROOF. Assume first that $i \neq 2d-1$. Consider the vector space $\overline{Y} = Y/\{0, \varepsilon\}$ with the basis $\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{2d-2}$ (images of $e_1, e_2, \ldots, e_{2d-2}$) and with the nonsingular symplectic form induced by (,). This is a vector space with a basis and form of the kind considered in [**4**, 3.1]. In particular, we can define a family of lagrangian subspaces $\mathfrak{F}(\overline{Y})$ of $\overline{Y}$ as in [**4**, 3.2]. It is clear that the inverse images under $Y \to \overline{Y}$ of the subspaces in $\mathfrak{F}(\overline{Y})$ are precisely the subspaces in $\mathfrak{F}(Y)$ which contain a basis element e_i $(i \leqslant 2d-2)$. By (2.2.3) all subspaces in $\mathfrak{F}(Y)$ are of this type. Therefore, in this case, our lemma follows from [**4**, Lemma 3.7].

The case where $i \neq 1$ is entirely similar: one considers the basis $\bar{e}_2, \ldots, \bar{e}_{2d-2}, \bar{e}_{2d-1}$ of $\bar{Y}$.

Since $d \geqslant 2$, one of the inequalities $i \neq 1$, $i \neq 2d-1$ must be satisfied and the lemma is proved.

2.4. Now let X be another vector space of dimension $2d-1$, and let $\langle\ ,\ \rangle$: $X \times Y \to F_2$ be a nonsingular bilinear pairing. We denote by e_i' the vector in X defined by $\langle e_i', y\rangle = (e_i, y)$, $y \in Y$, $1 \leqslant i \leqslant 2d-1$.

We have

PROPOSITION 2.5. *Let $x, y \to [x, y]$ be a map $X \times \tilde{Y} \to F_2$ with the following properties.*

(a) *For any $x \in X$ and any $C \in \mathfrak{F}(Y)$, the function $y \to [x, y]\,(C \to F_2)$ is F_2-linear.*

(b) *For any $x \in X$, $y \in \tilde{Y}$ and e_j such that $\langle x, e_j\rangle = 0$, we have*

$$(-1)^{[x,y]} + (-1)^{[x+e_j',y]} = (-1)^{\langle x,y\rangle} + (-1)^{\langle x+e_j',y\rangle}.$$

(c) *For any $y \in \tilde{Y}$, we have*

$$\sum_{x \in X} (-1)^{[x,y]} = \sum_{x \in X} (-1)^{\langle x,y\rangle}.$$

Then $[x, y] = \langle x, y\rangle$ for all $x \in X$, $y \in \tilde{Y}$.

The proof is almost identical to that of Proposition 3.8 in [**4**]. It uses Lemma 2.3 instead of [**4**, 3.7].

REMARK 2.6. We shall later apply this result in the following case. We take Y to be the set of all subsets of even cardinality of $Z_1 = \{0, 1, 2, \ldots, 2d-1\}$. It is an F_2-vector space with respect to the addition defined in $I, I' \to (I \cup I') - (I \cap I')$. The subsets $e_i = \{i-1, i\}$ $(1 \leqslant i \leqslant 2d-1)$ form a basis for Y. The symplectic form on Y is $I, I' \to |I \cap I'| \bmod 2$. If $I \in Y$, then the condition that $I \in \tilde{Y}$ is that $|I \cap \{0, 2, 4, \ldots\}| = |I \cap \{1, 3, 5, \ldots\}|$.

We take X to be the set of all subsets of $Z_1 = \{0, 1, \ldots, 2d-1\}$ (with the F_2-vector space structure defined just as for Y) taken modulo the subspace spanned by Z_1 itself. (Thus an element of X is an unordered pair $(I, Z_1 - I)$, where $I \subset Z_1$.) The duality pairing $\langle\ ,\ \rangle$: $X \times Y \to F_2$ is defined by $\langle I, J\rangle = |I \cap J| \bmod 2$. (Note that $|(Z_1 - I) \cap J| \equiv |I \cap J| \bmod 2$ whenever $|J| \equiv 0 \bmod 2$.)

The element $e_i' \in X$ is the set $\{i-1, i\}$ together with its complement in Z_1.

3. The main results.

3.1. Let k be an algebraic closure of the finite field F_q with q elements. Let V be a k-vector space of dimension $2n$ $(n \geqslant 1)$ endowed with a nonsingular quadratic form. Let G be the full orthogonal group of this quadratic form; it has two connected components. The identity component is denoted G^0. Let $e_1, f_1, e_2, f_2, \ldots, e_n, f_n$ be a basis of V with the following properties: the inner product of e_i with f_i is 1 $(1 \leqslant i \leqslant n)$, all other inner products of basis elements are zero and the quadratic form has value zero at all basis elements. Let F: $V \to V$ be the Frobenius semilinear map for an F_q-rational structure on V such that $F(e_i) = (e_i)$, $F(f_i) = f_i$ $(1 \leqslant i \leqslant n)$. We shall also denote by F: $G \to G$ the map satisfying $F(ge) = F(g)F(e)$ for all

$g \in G$, $e \in V$. This is the Frobenius map for an F_q-rational structure on G. Its fixed point set G^F is the full orthogonal group of an F_q-split quadratic form in $2n$ variables. Let $j: V \to V$ be the linear map which interchanges e_1 and f_1 and leaves the other basis vectors unchanged. Then $j \in G - G^0$, $j^2 = 1$. Let $F' = jF = Fj: V \to V$. We shall also denote by $F': G \to G$ the map satisfying $F'(ge) = F'(g)F'(e)$ for all $g \in G$, $e \in V$; thus, $F'(g) = jF'(g)j$ $(g \in G)$. This is the Frobenius map for an F_q-rational structure on G such that the fixed point set $G^{F'}$ is the full orthogonal group of a nonsplit quadratic form over F_q in $2n$ variables.

3.2. Let $\mathfrak{B}$ be the variety of all Borel subgroups of G. Then $F: \mathfrak{B} \to \mathfrak{B}$, $F': \mathfrak{B} \to \mathfrak{B}$ define two F_q-rational structures on $\mathfrak{B}$. We identify the Weyl group W of G^0 with the Coxeter group D_n; when $n \geq 2$, the simple reflections b, b' of D_n correspond to the two families of maximal isotropic subspaces in V. Thus, the orbits of G^0 on $\mathfrak{B} \times \mathfrak{B}$ are parametrized by the elements of D_n. Given $w \in D_n$, we denote by $\mathfrak{B}_w$ the variety of all $B \in \mathfrak{B}$ such that B, FB are in relative position w; the variety $\mathfrak{B}'_w$ is defined in the same way, using F' instead of F. The finite group $(G^0)^F$ (resp. $(G^0)^{F'}$) acts naturally, by conjugation, on $\mathfrak{B}_w$ (resp. on $\mathfrak{B}'_w$), and we thus have virtual representations

$$R_w = \sum_i (-1)^i H_c^i(\mathfrak{B}_w, \mathbf{Q}_l) \quad \text{of} \quad (G^0)^F$$

and

$$R'_w = \sum_i (-1)^i H_c^i(\mathfrak{B}'_w, \mathbf{Q}_l) \quad \text{of} \quad (G^0)^{F'}.$$

For each $M \in \mathfrak{R}(D_n)$, we define

$$R(M) = |D_n|^{-1} \sum_{w \in D_n} \operatorname{Tr}(w, M) R_w \in \mathfrak{R}\left((G^0)^F\right) \otimes \mathbf{Q}$$

(the Grothendieck group of virtual $\overline{\mathbf{Q}}_l$-representations of $(G^0)^F$, tensored with $\mathbf{Q}$). For each $M \in \mathfrak{R}(\hat{D}_n)^-$, we define

$$R(M) = |\hat{D}_n|^{-1} \sum_{w \in D_n} \operatorname{Tr}(w\varphi, M) R'_w \in \mathfrak{R}\left((G^0)^{F'}\right) \otimes \mathbf{Q}.$$

We have

PROPOSITION 3.3. *Let* $\underline{c} \in \mathfrak{R}(D_n)$ *be a virtual cell and let* $a = a(\underline{c})$ *(see* 1.18*). Then* $R(\underline{c})$ *is a linear combination with integral positive coefficients of irreducible representations of* $(G^0)^F$ *plus a linear combination with integral coefficients of elements* $R(E)$ *where* E *runs through the irreducible* $\mathbf{Q}[D_n]$*-modules such that* $a(E) > a$. *(For such* E, *we have* $\dim R(E) \equiv 0 \pmod{q^a}$.)

The proof is the same as that of Corollary 4.9 in [**4**], using 1.18 instead of [**4**, 2.20]. As in that proof, we must make use of the fact that $(G^0)^F$ can have unipotent cuspidal representations only for even values of n (cf. [**2**]).

3.4. We shall also need an analogue of Proposition 3.3 for the case of the twisted orthogonal group. This case is not covered by the arguments in Asai's paper [**1**]. We shall, instead, use Theorem 3.8 of [**6**] to deduce the following theorem.

THEOREM 3.5.

$$\sum(-1)^i H_c^i(\mathcal{B}'_w, \mathbf{Q}_l)^{(h)} = \frac{1}{2}\sum_E \operatorname{Tr}\big(T_w T_\varphi, \tilde{E}; h/2\big) R(E - \iota(E))$$

(sum over all irreducible, nondegenerate representations E of $\hat{D}_n$), as elements of $\mathcal{R}((G^0)^{F'}) \otimes \mathbf{Q}$. *(The superscript* $^{(h)}$ *denotes the part of weight* h.*)*

Just as in [**4**, 4.8] the previous theorem, together with [**4**, 4.5], shows that, for any $w \in D_n$ and any integer h, the element of $\mathcal{R}((G^0)^{F'}) \otimes \mathbf{Q}$ given by

$$\frac{(-1)^h}{2} \sum_{\substack{E \\ \text{nondeg.}}} \sum_{y \leqslant w} \sum_{j \geqslant 0} P_{y,w,j} \operatorname{Tr}\left(T_y T_\varphi, \tilde{E}; \frac{h}{2} - j\right) R(E - \iota(E))$$

is a linear combination with integral positive coefficients of irreducible representations of $(G^0)^{F'}$. Here $P_{y,w,j}$ are certain integers defined in [**4**, 4.4]. From this, and from 1.18, just as in [**4**, 4.9] we deduce

PROPOSITION 3.6. *Let $\underline{c} \in \mathcal{R}(\hat{D}_n)^-$ be a virtual cell and let $a = a(\underline{c})$ (see 1.18). Then $R(\underline{c})$ is a linear combination with integral positive coefficients of irreducible representations of $(G^0)^{F'}$ plus a linear combination with integral coefficients of elements $R(E - \iota(E))$ where E runs through the irreducible nondegenerate $\mathbf{Q}[\hat{D}_n]$-modules such that $a(E) > a$. (For such E, we have* $\dim R(E - \iota(E)) \equiv 0 \pmod{q^a}$.*)*

As in the proof of Corollary 4.9 in [**4**] we must make use of the fact that $(G^0)^{F'}$ can have unipotent cuspidal representations only in the case where the semisimple F_q-rank of $(G^0)^{F'}$ is even, i.e., n is odd (cf. [**2**]).

3.7. For each integer $1 \leqslant r \leqslant n-1$ we denote by $P_{r,s}$ $(s = n - r)$ the stabilizer in G^0 of the isotropic subspace spanned by $e_n, e_{n-1}, \ldots, e_{n-r+1}$. It is a parabolic subgroup of G^0, stable under F and F'. It has as Levi subgroup the group $L_{r,s}$ stabilizing the subspace spanned by $e_n, e_{n-1}, \ldots, e_{n-r+1}$ and the subspace spanned by $f_n, f_{n-1}, \ldots, f_{n-r+1}$. Note that $L_{r,s}$ is again stable under F and F' and it is isomorphic to the product of GL_r with a special orthogonal group in $2s$ variables which is split with respect to F and nonsplit with respect to F'.

Assume that notations have been chosen such that under the 1-1 correspondence between classes of maximal parabolics in G^0 and simple reflections in the Weyl group D_n of G^0 the stabilizer P_b of the subspace spanned by $e_n, e_{n-1}, \ldots, e_2, e_1$ corresponds to $b \in D_n$, while the stabilizer $P_{b'}$ of the subspace spanned by $e_n, e_{n-1}, \ldots, e_2, f_1$ corresponds to $b' \in D_n$. Then P_b, $P_{b'}$ are F-stable (but not F' stable) and admit F-stable Levi subgroups isomorphic to GL_n.

The unipotent representations of $(G^0)^F$ and $(G^0)^{F'}$ have been classified in [**2**] in terms of symbols of rank n and even defect.

Recall that a symbol of rank n and even defect is an unordered pair $\Lambda = \binom{T'}{T''}$ consisting of two finite subsets T', T'' of $\{0, 1, 2, 3, \ldots\}$ such that $|T'| + |T''| = 2m$, $\sum_{\lambda \in T'} \lambda + \sum_{\mu \in T''} \mu = n + m^2 - m$. (The defect of Λ is $||T'| - |T''||$.)

There is an equivalence relation on such pairs generated by the shift

$$\binom{T'}{T''} \sim \binom{0 \amalg (T'+1)}{0 \amalg (T''+1)}$$

and we shall often identify a symbol with its equivalence class. This notion contains as a special case the notion of special symbol of rank n and defect 0 considered in 1.9. A symbol such that $T' \neq T''$ is said to be nondegenerate. A symbol such that $T' = T''$ is said to be degenerate; it is then automatically special. Any symbol Λ of rank n and even defect gives rise to a special symbol Z of rank n and defect 0, as follows. We take the entries of Λ and arrange them in a single row in increasing order. We get a monotonic sequence in which there may be equalities but no two consecutive equalities. The first, third, fifth, etc., term of this sequence will be the second row of Z while the second, fourth, etc., term of this will be the first row of Z. Thus Z is the unique special symbol of defect 0 whose set of entries (some of which may be repeated twice) coincides with the set of entries of Λ (union of T' and T'', with common elements repeated twice). We shall then set $a_\Lambda = a([Z])$, if Z is nondegenerate, $a_\Lambda = a([Z]_b) = a([Z]_{b'})$ if Z is degenerate. In the case where Λ has rank n and defect 0, we have $a_\Lambda = a[\Lambda]$ (see 1.6). The following lemma is a consequence of the results in 1.7, (1.4.1), [**4**, 2.4, 2.7], and the results in [**2**] on classification and degrees of unipotent representations. (Compare Lemma 5.2 in [**4**].)

LEMMA 3.8. *There exists a* 1-1 *correspondence* $\Lambda \leftrightarrow \rho(\Lambda)$ *between the set of symbols of rank n and defect* $\equiv 2 \pmod 4$ (*up to shift*) *and the set of unipotent representations* (*up to isomorphism*) *of* $(G^0)^{F'}$, *with the properties* (i), (ii), (iii) *below.*

There exists a correspondence $\Lambda \leftrightarrow \rho(\Lambda)$ *between the set of symbols of rank n and defect* $\equiv 0 \pmod 4$ (*up to shift*) *and the set of unipotent representations* (*up to isomorphism*) *of* $(G^0)^F$, *which is* 1-1, *except that to each degenerate symbol* Λ (*of defect* 0) *there correspond two unipotent representations* $\rho(\Lambda)_b$, $\rho(\Lambda)_{b'}$, *and which has the properties* (i), (ii), (iii) *below.*

(i) *If Z is the special symbol corresponding to* Λ (*of rank n and even defect*), $a = a_\Lambda$ *and* $d = d[Z]$ *is such that* $2d$ *is the number of singles of Z, then*

$2^{d-1}\dim \rho(\Lambda) \equiv q^a \pmod{q^{a+1}}$ *if* Λ *is nondegenerate*, $\dim \rho(\Lambda)_b = \dim \rho(\Lambda)_{b'} \equiv q^a \pmod{q^{a+1}}$ *if* Λ *is degenerate.*

(ii) *Let* Λ *be a symbol of rank n and even defect. Let t be an integer,* $t \geq$ *all entries in* Λ. *Let* $\bar{T}' = \{t - i \mid 0 \leq i \leq t,\ i \notin T''\}$, $\bar{T}'' = \{t - i \mid 0 \leq i \leq t,\ i \notin T'\}$, *and let* $\bar{\Lambda} = \binom{\bar{T}'}{\bar{T}''}$. *This is again a symbol of rank n, even defect and*

$$D(\rho(\Lambda)) = \rho(\bar{\Lambda}) \quad \textit{if } \Lambda \textit{ is nondegenerate},$$

$$\left.\begin{aligned} D(\rho(\Lambda)_b) &= \rho(\bar{\Lambda})_b \textit{ or } \rho(\bar{\Lambda})_{b'} \\ D(\rho(\Lambda)_{b'}) &= \rho(\bar{\Lambda})_b \textit{ or } \rho(\bar{\Lambda})_{b'} \end{aligned}\right\} \quad \textit{if } \Lambda \textit{ is degenerate},$$

where D is the duality operator defined as in [**4**, (4.6.5)] (*but sum only over those P which are defined over* F_q).

(iii) *Let* Λ' *be a symbol of rank s and even defect* $(1 \leq s \leq n-1)$ *and let* $r = n - s$; *we may assume that* Λ' *has* $\geq r$ *entries. We associate to* Λ' *a symbol* Λ (*or two symbols* Λ_{I}, Λ_{II}) *of rank n and same defect as* Λ', *by increasing by* 1 *each of the r largest entries* *in* Λ' (*as in* 1.7, *where the case of symbols of defect* 0 *was considered*). *If* Λ' *is nondegenerate of defect* $\equiv 0 \pmod 4$ *then*

$$\mathrm{Ind}_{P^F_{r,s}}^{(G^0)^F}(\mathrm{St}_r \otimes \rho(\Lambda')) = \begin{cases} \rho(\Lambda) + \tau & \text{or} \\ \rho(\Lambda_{\mathrm{I}}) + \rho(\Lambda_{\mathrm{II}}) + \tau; \end{cases}$$

if Λ' *is of defect* $\equiv 2 \pmod 4$ *then*

$$\mathrm{Ind}_{P^{F'}_{r,s}}^{(G^0)^{F'}}(\mathrm{St}_r \otimes \rho(\Lambda')) = \begin{cases} \rho(\Lambda) + \tau & \text{or} \\ \rho(\Lambda_{\mathrm{I}}) + \rho(\Lambda_{\mathrm{II}}) + \tau; \end{cases}$$

if Λ *is degenerate, and r is odd, then*

$$\mathrm{Ind}_{P^F_{r,s}}^{(G^0)^F}(\mathrm{St}_r \otimes \rho(\Lambda')_b) = \mathrm{Ind}_{P^F_{r,s}}^{(G^0)^F}(\mathrm{St}_r \otimes \rho(\Lambda')) = \rho(\Lambda);$$

if Λ *is degenerate and r is even, then*

$$\mathrm{Ind}_{P^F_{r,s}}^{(G^0)^F}(\mathrm{St}_r \otimes \rho(\Lambda')_b) = \rho(\Lambda)_b$$

and the same holds with b replaced by b′.

Here, τ *is a* **Z**-*linear combination of representations* $\rho(\Lambda_i)$ *such that* $a_{\Lambda_i} > a_\Lambda$ (*or* $a_{\Lambda_i} > a_{\Lambda_{\mathrm{I}}} = a_{\Lambda_{\mathrm{II}}}$) *and* St_r *is the Steinberg representation of* $GL_n(F_q)$.

Moreover,

$$\mathrm{Ind}_{P^F_b}^{(G^0)^F}(\mathrm{St}_n) = \rho\begin{pmatrix} 1,2,3,\ldots,\frac{n}{2} \\ 1,2,3,\ldots,\frac{n}{2} \end{pmatrix}_b + \tau \quad \text{if } n \text{ is even},$$

$$= \rho\begin{pmatrix} 1,2,3,\ldots,\frac{n+1}{2} \\ 0,2,3,\ldots,\frac{n+1}{2} \end{pmatrix} + \tau \quad \text{if } n \text{ is odd},$$

and the same holds with b replaced by b′. Here τ *is a* **Z**-*linear combination of representations* $\rho(\Lambda_i)$ *such that* $a_{\Lambda_i} > \frac{1}{2}n(n-1)$.

REMARK 3.9. The unipotent representations of $(G^0)^F$ or $(G^0)^{F'}$ of form $\rho(\Lambda)$, Λ nondegenerate, are said to be nondegenerate. The other unipotent representations (which can only exist for $(G^0)^F$) are said to be degenerate. We have

LEMMA 3.10. *For any unipotent representation* ρ *of* $(G^0)^F$ *or of* $(G^0)^{F'}$ *there exist integers* $d = d(\rho) \geq 0$, $a = a(\rho) \geq 0$ *such that*

$$\left.\begin{array}{ll} (3.10.1) & 1 \leq d^2 \leq n \\ (3.10.2) & 2^{d-1}\dim(\rho) \equiv q^a \pmod{q^{a+1}} \end{array}\right\} \text{ if } \rho \text{ is nondegenerate},$$

$$\left.\begin{array}{ll} (3.10.3) & d = 0 \\ (3.10.4) & \dim(\rho) \equiv q^a \pmod{q^{a+1}} \end{array}\right\} \text{ if } \rho \text{ is degenerate}.$$

Let $\tilde{\sigma}(n)$ *be the largest integer such that* $\tilde{\sigma}(n)^2 \leq n$. *If* $q > 2^{\tilde{\sigma}(n)-1}$ *then the conditions* (3.10.1), (3.10.2), (3.10.3), (3.10.4) *determine* $d(\rho)$, $a(\rho)$ *uniquely.*

The proof is the same as that of Lemma 5.3 in [**4**].

LEMMA 3.11. *If $\underline{c}$ is a virtual cell in $\mathfrak{R}(D_n)$ or $\mathfrak{R}(D_n)^-$, then*

$$\dim R(\underline{c}) \equiv q^{a(\underline{c})} \pmod{q^{a(\underline{c})+1}}.$$

This follows from [2, (2.6.2)]. (Compare Lemma 5.4 in [4].)

LEMMA 3.12. (i) *Let Z be a nondegenerate symbol of rank n and defect* 0, *with $2d$ singles, let Φ be an admissible arrangement for Z and let $\hat{\Phi}$, $\hat{\Phi}'$ be two subsets of Φ such that $|\hat{\Phi}| \equiv |\hat{\Phi}'| \pmod 2$. Let $\underline{c} = \underline{c}(Z, \Phi, \Phi)$, $\underline{c}' = \underline{c}(Z, \Phi, \Phi')$. Then*

$$\langle R(\underline{c}), R(\underline{c}')\rangle = \begin{cases} 2^{d-1}, & \text{if } \hat{\Phi} = \hat{\Phi}', \\ 0, & \text{if } \hat{\Phi} = \hat{\Phi}' \end{cases}$$

(*inner product over $(G^0)^F$ if $|\hat{\Phi}|$ is even and over $(G^0)^{F'}$ if $|\hat{\Phi}|$ is odd*).

(ii) *If Z is a degenerate symbol of rank n and defect* 0 *then*

$$\langle R([Z]_b), R([Z]_b)\rangle = \langle R([Z]_{b'}), R([Z]_{b'})\rangle = 1$$

(*inner product over $(G^0)^F$*).

The proof is similar to that of Lemma 5.5 in [4].

PROPOSITION 3.13. *Let $\underline{c}$ be a virtual cell in $\mathfrak{R}(D_n)$ or $\mathfrak{R}(D_n)^-$ and let Z be the corresponding special symbol. Let $a = a(\underline{c})$ be defined as in* 1.18 *and let $d = d(\underline{c})$ be such that $2d$ is the number of singles in $\underline{c}$. Assume that $q > 2^{2\tilde{\sigma}(n)-1}$ ($\tilde{\sigma}(n)$ as in Lemma* 3.10). *If Z is degenerate, then $R(\underline{c})$ is an irreducible unipotent representation ρ of $(G^0)^F$ satisfying $a(\rho) = a$, $d(\rho) = d = 0$. If Z is nondegenerate, then*

$$R(c) = \sum_{i=1}^{2^{d-1}} \rho_i$$

where ρ_i $(1 \leqslant i \leqslant 2^{d-1})$ are distinct irreducible representations of $(G^0)^F$ (if $\underline{c} \in \mathfrak{R}(D_n)$) or of $(G^0)^{F'}$ (if $\underline{c} \in \mathfrak{R}(D_n)^-$) satisfying $a(\rho_i) = a$, $d(\rho_i) = d$.

The proof is similar to that of Theorem 5.6 in [4].

3.14. Let Z be a tableau as in 1.9, and let Z_1 be the set of singles of Z. We can write $Z_1 = Z_1^* \amalg (Z_1)_*$ where Z_1^* (resp. $(Z_1)_*$) is the set of entries of Z_1 appearing in the first row (resp. second row) of Z. Let $d = |Z_1^*| = |(Z_1)_*|$. Let Z_2 be the set of elements which appear in both rows of Z. Thus,

$$Z = \begin{pmatrix} Z_2 \amalg Z_1^* \\ Z_2 \amalg (Z_1)_* \end{pmatrix}.$$

THEOREM 3.15. *Assume that $q > 2^{2\tilde{\sigma}(n)-1}$ ($\tilde{\sigma}(n)$ as in Lemma* 3.10).

(i) *If Z is degenerate then*

$$R[Z]_b = \rho(Z)_b, \qquad \rho[Z]_{b'} = \rho(Z)_{b'}.$$

(ii) *Assume now that Z is nondegenerate, so that Z_1 is nonempty. For each subset $M \subset Z_1$ we denote $M^{\#} = M \cup (Z_1)_* - (M \cap (Z_1)_*)$. If $|M'| \equiv d \pmod 2$, $|M| = d$, then*:

$$\left\langle \rho\begin{pmatrix} Z_2 \amalg (Z_1 - M') \\ Z_2 \amalg M' \end{pmatrix}, R\begin{bmatrix} Z_2 \amalg (Z_1 - M) \\ Z_2 \amalg M \end{bmatrix} \right\rangle_{(G^0)^F} = (-1)^{|M^\# \cap M'^\#|} 2^{-(d-1)}. \tag{3.15.1}$$

If $|M'| \not\equiv d \pmod 2$, $|M| = d$, *then*:

$$\left\langle \rho\begin{pmatrix} Z_2 \amalg (Z_1 - M') \\ Z_2 \amalg M' \end{pmatrix}, R\left(\begin{bmatrix} Z_2 \amalg (Z_1 - M) \\ Z_2 \amalg M \end{bmatrix} - \begin{bmatrix} Z_2 \amalg M \\ Z_2 \amalg (Z_1 - M) \end{bmatrix}\right) \right\rangle_{(G^0)^{F'}} = (-1)^{|M^\# \cap M'^\#|} 2^{-(d-1)}. \tag{3.15.2}$$

COROLLARY 3.16. (i) *Let* $w \in D_n$. *With the assumptions of* 3.15(i), *we have*

$$\langle \rho(Z)_b, R_w \rangle_{(G^0)^F} = \operatorname{Tr}(w, [Z]_b).$$

The same holds with b replaced by b'.

(ii) *We preserve the assumptions of* 3.15(ii). *Let* $w \in D_n$. *If* $M' \subset Z_1$, $|M'| \equiv d \pmod 2$, *then*

$$\left\langle \rho\begin{pmatrix} Z_2 \amalg (Z_1 - M') \\ Z_2 \amalg M' \end{pmatrix}, R_w \right\rangle_{(G^0)^F} = 2^{-d} \sum_{\substack{M \subset Z_1 \\ |M| = d}} (-1)^{|M^\# \cap M'^\#|} \operatorname{Tr}\left(w, \begin{bmatrix} Z_2 \amalg (Z_1 - M) \\ Z_2 \amalg M \end{bmatrix}\right).$$

If $M' \subset Z_1$, $|M'| \not\equiv d \pmod 2$, *then*

$$\left\langle \rho\begin{pmatrix} Z_2 \amalg (Z_1 - M') \\ Z_2 \amalg M' \end{pmatrix}, R'_w \right\rangle_{(G^0)^{F'}} = 2^{-d} \sum_{\substack{M \subset Z_1 \\ |M| = d}} (-1)^{|M^\# \cap M'^\#|} \operatorname{Tr}\left(w\varphi, \begin{bmatrix} Z_2 \amalg (Z_1 - M) \\ Z_2 \amalg M \end{bmatrix}\right).$$

3.17. PROOF OF THEOREM 3.15. As in the proof of Theorem 5.8 in [**4**], we may reduce the general case to the case where

$$Z = Z_1 = \begin{pmatrix} 1, 3, 5, \ldots, 2d-1 \\ 0, 2, 4, \ldots, 2d-2 \end{pmatrix}, \qquad d \geqslant 1.$$

See [**4**, 5.10 A), B)]. In this special case, we see as in [**4**, 5.10 C)] that the left-hand side of (3.15.1) must be of the form $(-1)^{[M'^\#, M^\#]} 2^{-(d-1)}$ and the left-hand side of (3.15.2) must be of the form $(-1)^{[M'^\#, M^\#]} 2^{-(d-1)}$, where $[M'^\#, M^\#]$ is a certain integer modulo 2. Let X, Y, $\tilde{Y}$ be defined as in 2.6. Then $M'^\#, M^\# \to [M'^\#, M^\#]$ is a map $[\ ,\ ]\colon X \times \tilde{Y} \to F_2$. As in [**4**, 5.10 C)] we see that this map satisfies the assumptions of Proposition 2.5. It follows that $[M'^\#, M^\#] \equiv |M'^\# \cap M^\#| \bmod 2$ for all $M'^\# \in X$, $M^\# \in \tilde{Y}$, and the theorem is proved.

3.18. For each unipotent representation ρ of $(G^0)^F$ there is a well-defined sign $\lambda_\rho = \pm 1$ such that, whenever ρ is contained in a generalized eigenspace of F: $H_c^i(X_w) \to H_c^i(X_w)$, the corresponding eigenvalue of F is of the form $\lambda_\rho \cdot q^k$, where k is an integer. If ρ is degenerate, then $\lambda_\rho = 1$. If ρ is nondegenerate, of form

$$\rho = \rho\begin{pmatrix} Z_2 \amalg (Z_1 - M') \\ Z_2 \amalg M' \end{pmatrix}$$

(as in 3.14, 3.15(ii)), with $|Z_1| = 2d > 0$, $M' \subset Z_1$, $|M'| \equiv d \pmod 2$, then $\lambda_\rho = (-1)^{(|M'|-d)/2}$. This is proved in the same way as Proposition 6.6 in **[4]**.

3.19. It is very likely that 3.15 and 3.16 hold without restrictions on q. Indeed, T. Asai has recently shown (*Unipotent characters of* Sp_{2n} *and* SO_{2n+1} *over* F_q *with small* q) that the main result of **[4]** holds also for small q. No doubt, the same method should apply in the case of even orthogonal groups as well.

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DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASSACHUSETTS 02139